Infinite Sequences and Series

Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

   (b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

   (c) The terms a_n become large as n becomes large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

   (b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \ldots\}$.

4. $a_n = \dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \ldots\right\} = \left\{1, \dfrac{3}{5}, \dfrac{1}{2}, \dfrac{5}{11}, \dfrac{3}{7}, \ldots\right\}$.

5. $a_n = \dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \ldots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \ldots\right\}$.

6. $a_n = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$, so the sequence is
   $\{2, 2\cdot 4, 2\cdot 4\cdot 6, 2\cdot 4\cdot 6\cdot 8, 2\cdot 4\cdot 6\cdot 8\cdot 10, \ldots\} = \{2, 8, 48, 384, 3840, \ldots\}$.

7. $a_n = \sin \dfrac{n\pi}{2}$, so the sequence is $\{1, 0, -1, 0, 1, \ldots\}$.

8. $a_1 = 1$, $a_{n+1} = \dfrac{1}{1+a_n}$, so the sequence is

$$\left\{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{2}}, \frac{1}{1+\frac{2}{3}}, \frac{1}{1+\frac{3}{5}}, \ldots\right\} = \left\{1, \frac{1}{2}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{5}{3}}, \frac{1}{\frac{8}{5}}\right\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots\right\}$$

9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \dfrac{1}{2n}$.

11. $\{2, 7, 12, 17, \ldots\}$. Each term is larger than the preceding one by 5, so
    $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\left\{-\dfrac{1}{4}, \dfrac{2}{9}, -\dfrac{3}{16}, \dfrac{4}{25}, \ldots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

14. $\{0, 2, 0, 2, 0, 2, \ldots\}$. One is halfway between 0 and 2, so we can think of alternately subtracting and adding 1 (from 1 and to 1) to obtain the given sequence: $a_n = 1 - (-1)^{n-1}$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot 0 = 0$ by (7) with $r = \frac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

21. $a_n = \dfrac{(-1)^{n-1}n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 5. Converges

22. $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, \ldots\}$. This sequence oscillates among 1, 0, and -1, so the sequence diverges.

23. $a_n = 2 + \cos n\pi$, so

$$\{a_n\} = \{2+\cos\pi, 2+\cos 2\pi, 2+\cos 3\pi, 2+\cos 4\pi, \ldots\} = \{2-1, 2+1, 2-1, 2+1, \ldots\}$$
$$= \{1, 3, 1, 3, \ldots\}$$

This sequence oscillates between 1 and 3, so it diverges.

24. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Convergent

25. $0 < \dfrac{3+(-1)^n}{n^2} \le \dfrac{4}{n^2}$ and $\lim\limits_{n\to\infty} \dfrac{4}{n^2} = 0$, so $\left\{\dfrac{3+(-1)^n}{n^2}\right\}$ converges to 0 by the Squeeze Theorem.

26. $\lim\limits_{n\to\infty} \dfrac{n!}{(n+2)!} = \lim\limits_{n\to\infty} \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{1\cdot 2\cdot 3\cdot\cdots\cdot n\,(n+1)\,(n+2)} = \lim\limits_{n\to\infty} \dfrac{1}{(n+2)\,(n+1)} = 0$. Convergent

27. $\lim\limits_{x\to\infty} \dfrac{\ln(x^2)}{x} = \lim\limits_{x\to\infty} \dfrac{2\ln x}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{2/x}{1} = 0$, so by Theorem 2, $\left\{\dfrac{\ln(n^2)}{n}\right\}$ converges to 0.

28. $\lim\limits_{n\to\infty} \sin\left(\dfrac{1}{n}\right) = \sin 0 = 0$ since $\dfrac{1}{n} \to 0$ as $n \to \infty$, so by Theorem 5, $\left\{(-1)^n \sin\left(\dfrac{1}{n}\right)\right\}$ converges to 0.

29. $b_n = \sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n}\right)\dfrac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \dfrac{2}{\sqrt{n+2}+\sqrt{n}} < \dfrac{2}{2\sqrt{n}} = \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. So by the Squeeze Theorem with $a_n = 0$ and $c_n = 1/\sqrt{n}$, $\left\{\sqrt{n+2} - \sqrt{n}\right\}$ converges to 0.

30. $\lim\limits_{x\to\infty} \dfrac{\ln(2+e^x)}{3x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x/(2+e^x)}{3} = \lim\limits_{x\to\infty} \dfrac{1}{6e^{-x}+3} = \dfrac{1}{3}$, so by Theorem 2, $\lim\limits_{n\to\infty} \dfrac{\ln(2+e^n)}{3n} = \dfrac{1}{3}$. Convergent

31. $\lim\limits_{x\to\infty} \dfrac{x}{2^x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1}{(\ln 2)\,2^x} = 0$, so by Theorem 2, $\{n2^{-n}\}$ converges to 0.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1+\dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Convergent

33. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

34. $y = (1+3x)^{1/x} \quad\Rightarrow\quad \ln(y) = \dfrac{1}{x}\ln(1+3x) \quad\Rightarrow\quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln(1+3x)}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{3/(1+3x)}{1} = 0 \quad\Rightarrow$ $\lim\limits_{x\to\infty} y = e^0 = 1$, so by Theorem 2, $\left\{(1+3n)^{1/n}\right\}$ converges to 1.

35. The series converges, since

$$a_n = \frac{1+2+3+\cdots+n}{n^2} = \frac{n(n+1)/2}{n^2} \text{ [sum of the first } n \text{ positive integers]} = \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}$$

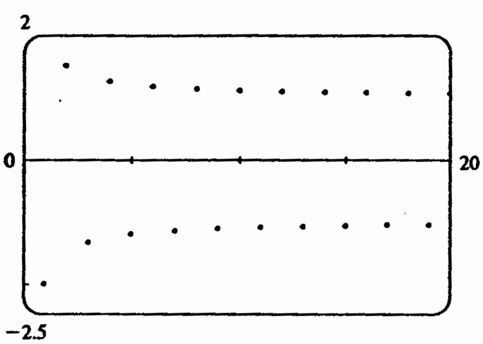
as $n \to \infty$.

36. $0 \le |a_n| = \dfrac{n\,|\cos n|}{n^2+1} \le \dfrac{n}{n^2+1} = \dfrac{1}{n+1/n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\{a_n\}$ converges to 0.

37. $a_n = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \ge \dfrac{1}{2}\cdot\dfrac{n}{2} = \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

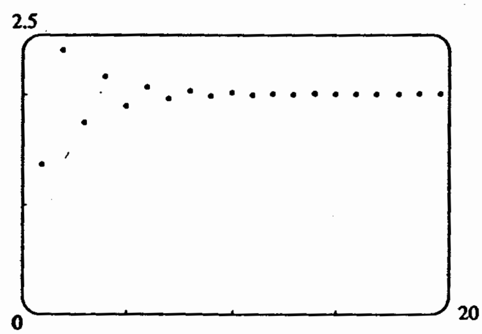
38. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \le 3\cdot\dfrac{3}{2}\cdot\dfrac{3}{n} = \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\left\{(-3)^n/n\right\}$ converges to 0.

39.



From the graph, we see that the sequence $\left\{(-1)^n \dfrac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

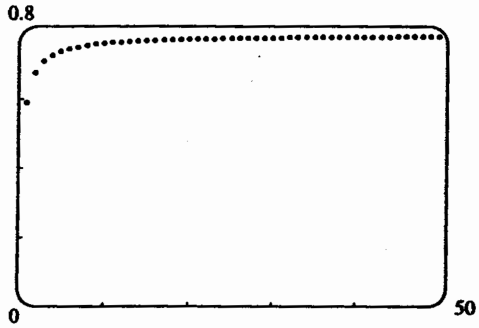
40.



From the graph, it appears that the sequence converges to 2.

$\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (7), and hence $\left\{2+\left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2+0=2$.

41.

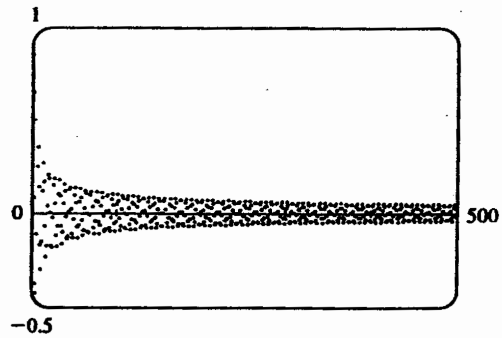


From the graph, it appears that the sequence converges to about 0.78.

$$\lim_{n\to\infty} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{2}{2+1/n} = 1, \text{ so}$$

$$\lim_{n\to\infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}.$$

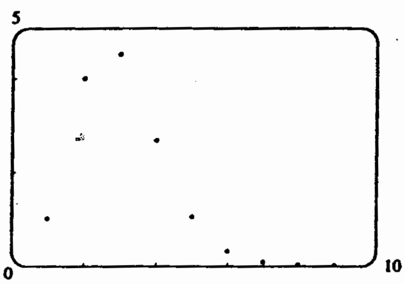
42.



From the graph, it appears that the sequence converges (slowly) to 0.

$0 \le \dfrac{|\sin n|}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

43.



From the graph, it appears that the sequence converges to 0.

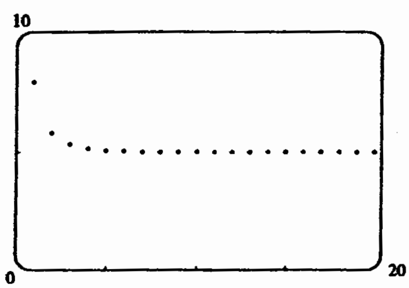
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \cdots \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad (\text{for } n \ge 4)$$

$$= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

44.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\sqrt[n]{5^n}$$

$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty$$

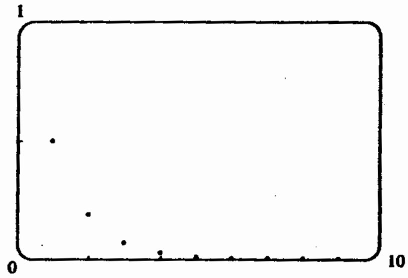
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5$$

so $\lim\limits_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

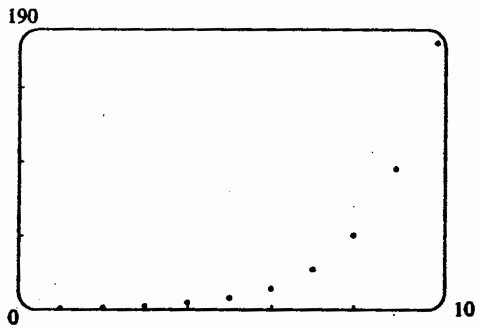
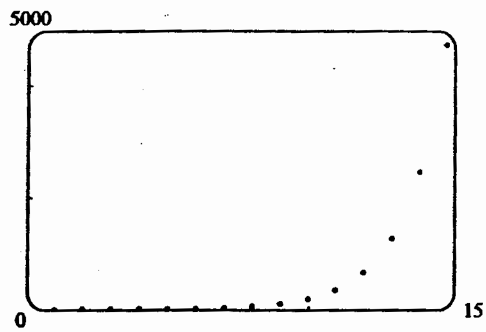
45. 

From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} \right\}$ converges to 0.

46.  

From the graphs, it seems that the sequence diverges. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\frac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \frac{2n+1}{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{2n+1}{n+1}$ (induction hypothesis). But $\frac{2n+1}{n+1} \geq \frac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \Leftrightarrow 4n+2 \geq 3n+3 \Leftrightarrow n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{ \left(\frac{3}{2}\right)^{n-1} \right\}$ diverges (by Equation 7), so does the given sequence $\{a_n\}$.

47. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060, a_2 = 1123.60, a_3 = 1191.02, a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n\to\infty} a_n = 1000 \lim_{n\to\infty} (1.06)^n$, so the sequence diverges by (7) with $r = 1.06 > 1$.

48. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

49. If $|r| \geq 1$, then $\{r^n\}$ diverges by (7), so $\{nr^n\}$ diverges also, since $|nr^n| = n|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim_{x\to\infty} xr^x = \lim_{x\to\infty} \frac{x}{r^{-x}} \overset{H}{=} \lim_{x\to\infty} \frac{1}{(-\ln r)r^{-x}} = \lim_{x\to\infty} \frac{r^x}{-\ln r} = 0$, so $\lim_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

50. (a) Let $\lim_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \Leftrightarrow n > N - 1$. It follows that $\lim_{n\to\infty} a_{n+1} = L$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

(b) If $L = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow$ $L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non-negative if it exists).

51. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent, that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

52. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

53. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

54. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$,

and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

55. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

56. $a_n = 3 + (-1)^n/n$ defines a sequence that is not monotonic. The first few terms are $2, 3.5, 2.\overline{6}, 3.25$, and 2.8, showing that the sequence is neither increasing nor decreasing. The sequence is bounded since $2 \leq a_n \leq 3.5$ for all $n \geq 1$.

57. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$,

$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

58. $a_n = \dfrac{\sqrt{n}}{n+2}$ defines a sequence that is neither increasing nor decreasing since $a_1 < a_2$ and $a_2 > a_3$. ($a_1 = \frac{1}{3} = 0.\overline{3}$, $a_2 = \frac{\sqrt{2}}{4} \approx 0.354$, and $a_3 = \frac{\sqrt{3}}{5} \approx 0.346$.) But the sequence $\{a_n \mid n \geq 2\}$ obtained by omitting the first term a_1 *is* decreasing. To see this, note that if $f(x) = \dfrac{\sqrt{x}}{x+2}$ for $x \geq 0$, then

$f'(x) = \dfrac{\frac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2} = \dfrac{(x+2) - 2x}{2\sqrt{x}(x+2)^2} = \dfrac{2-x}{2\sqrt{x}(x+2)^2} \leq 0$ for $x \geq 2$. The sequence is bounded since $a_n > 0$ for all $n \geq 1$ and $a_n \leq a_2 = \frac{\sqrt{2}}{4}$ for all $n \geq 1$.

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

   (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

10. (a) Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

    (b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$ is a geometric series with $a = 4$ and $r = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to $\frac{4}{1-2/5} = \frac{4}{3/5} = \frac{20}{3}$.

12. $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is a geometric series with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n \Rightarrow a = \frac{1}{e^2} = |r| < 1$, so the series converges to $\frac{1/e^2}{1 - 1/e^2} = \frac{1}{e^2 - 1}$.

19. For $\sum_{n=1}^{\infty} 3^{-n} 8^{n+1} = \sum_{n=1}^{\infty} 8\left(\frac{8}{3}\right)^n$, $a = \frac{64}{3}$ and $r = \frac{8}{3} > 1$, so the series diverges.

20. $\sum_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n \Rightarrow a = 4$, $|r| = \frac{4}{5} < 1$, so the series converges to $\frac{4}{1-4/5} = 20$.

21. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty} (3/n) = 3\sum_{n=1}^{\infty}(1/n)$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty}(1/n)$, which diverges. [If $\sum_{n=1}^{\infty}(3/n)$ were to converge, then $\sum_{n=1}^{\infty}(1/n)$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Converges. $s_n = \sum_{i=1}^{n} \frac{1}{i(i+2)} = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+2}\right)$ (using partial fractions) $= \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+2}\right)$. The latter sum is a telescoping series:

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty}\left(1+\frac{1}{n^2+2n}\right) = 1 \neq 0.$$

25. $\sum_{n=1}^{\infty}\left[2(0.1)^n + (0.2)^n\right] = 2\sum_{n=1}^{\infty}(0.1)^n + \sum_{n=1}^{\infty}(0.2)^n$. These are convergent geometric series and so by Theorem 8, their sum is also convergent. $2\left(\frac{0.1}{1-0.1}\right) + \frac{0.2}{1-0.2} = \frac{2}{9} + \frac{1}{4} = \frac{17}{36}$

26. Converges. $s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n}\left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$$\left(\tfrac{1}{2}-\tfrac{1}{4}\right) + \left(\tfrac{1}{3}-\tfrac{1}{5}\right) + \left(\tfrac{1}{4}-\tfrac{1}{6}\right) + \left(\tfrac{1}{5}-\tfrac{1}{7}\right) + \cdots + \left(\tfrac{1}{n}-\tfrac{1}{n+2}\right) + \left(\tfrac{1}{n+1}-\tfrac{1}{n+3}\right) = \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{n+2} - \tfrac{1}{n+3}$$

(telescoping series). Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.$

27. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\sqrt{1+n^2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$, so the series diverges by the Test for Divergence.

28. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} + 2\sum_{n=1}^{\infty}\frac{1}{3^{n-1}} = \frac{1}{1-1/2} + 2\left(\frac{1}{1-1/3}\right) = 5$

29. Converges. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n}+\frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

30. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty}\left(\frac{1}{2+5/n}\right) = \ln\tfrac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\arctan n = \tfrac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{5+2^{-n}} = \frac{1}{5} \neq 0$, so the series diverges by the Test for Divergence.

33. $s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + [\ln n - \ln(n+1)] = \ln 1 - \ln(n+1) = -\ln(n+1)$ (telescoping series). Thus, $\lim_{n\to\infty} s_n = -\infty$, so the series is divergent.

34. $s_n = \displaystyle\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)} = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1}{i+1} + \frac{1/2}{i+2}\right) = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+1}\right) + \sum_{i=1}^{n}\left(-\frac{1/2}{i+1} + \frac{1/2}{i+2}\right),$ both of which are clearly telescoping sums, so

$$s_n = \left[\frac{1}{2} - \frac{1}{2(n+1)}\right] + \left[-\frac{1}{4} + \frac{1}{2(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}.$

35. $0.\overline{2} = \dfrac{2}{10} + \dfrac{2}{10^2} + \cdots = \dfrac{2/10}{1-1/10} = \dfrac{2}{9}$

36. $0.\overline{73} = \dfrac{73}{10^2} + \dfrac{73}{10^4} + \cdots = \dfrac{73/10^2}{1-1/10^2} = \dfrac{73/100}{99/100} = \dfrac{73}{99}$

37. $3.\overline{417} = 3 + \dfrac{417}{10^3} + \dfrac{417}{10^6} + \cdots = 3 + \dfrac{417/10^3}{1-1/10^3} = 3 + \dfrac{417}{999} = \dfrac{3414}{999} = \dfrac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \dfrac{54}{10^3} + \dfrac{54}{10^5} + \cdots = 6.2 + \dfrac{54/10^3}{1 - 1/10^2} = \dfrac{62}{10} + \dfrac{54}{990} = \dfrac{6192}{990} = \dfrac{344}{55}$

39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1 - 0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{8^4} + \cdots = 5 + \dfrac{6021/10^4}{1 - 1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$. In that case, the sum of the series is $\dfrac{x/3}{1 - x/3} = \dfrac{x}{3 - x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x - 4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x - 4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4\,|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\dfrac{1}{1-(x+3)/2} = \dfrac{2}{2-(x+3)} = -\dfrac{2}{x+1}$.

45. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$ is geometric with $r = \dfrac{1}{x}$, so it converges whenever $\left|\dfrac{1}{x}\right| < 1$ $\Leftrightarrow$ $|x| > 1$ $\Leftrightarrow$ $x > 1$ or $x < -1$, and the sum is $\dfrac{1}{1 - 1/x} = \dfrac{x}{x-1}$.

46. $\sum_{n=0}^{\infty} \tan^n x$ is geometric and converges when $|\tan x| < 1$ $\Leftrightarrow$ $-1 < \tan x < 1$ $\Leftrightarrow$ $n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4}$ (n any integer). On these intervals the sum is $\dfrac{1}{1 - \tan x}$.